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## On the Dynamics of a Class of Kolmogorov Systems

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*In this paper we characterize the integrability and the non-existence of limit cycles of Kolmogorov systems of the form*

$$\begin{cases} x' = x \left( P(x, y) + \sqrt{R(x, y)} \right), \\ y' = y \left( Q(x, y) + \sqrt{R(x, y)} \right), \end{cases}$$

where  $P(x, y)$ ,  $Q(x, y)$ ,  $R(x, y)$ , homogeneous polynomials of degree  $n$ ,  $n$ ,  $m$ , respectively.

*Keywords:* Kolmogorov system, first integral, periodic orbits, limit cycle.

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## Introduction

The autonomous differential system on the plane given by

$$\begin{cases} x' = \frac{dx}{dt} = xF(x, y), \\ y' = \frac{dy}{dt} = yG(x, y), \end{cases} \quad (1)$$

known as Kolmogorov system, the derivatives are performed with respect to the time variable and  $F$ ,  $G$  are two functions in the variables  $x$  and  $y$ . If  $F$  and  $G$  are linear (Lotka-Volterra-Gause model), then it is well known that there is at most one critical point in the interior of the realistic quadrant ( $x \geq 0, y \geq 0$ ), and there are no limit cycles [10, 14, 16]. There are many natural phenomena which can be modeled by the Kolmogorov systems such as mathematical ecology and population dynamics [12, 17, 18] chemical reactions, plasma physics [13], hydrodynamics [5], etc.

In the qualitative theory of planar dynamical systems [7–9, 15], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem. There is a huge literature about the limit cycles, most of it deals essentially with their detection, number and stability [1–4, 11]. We recall that in the phase plane, a limit cycle of system (1) is an isolated periodic orbit in the set of all periodic orbits of system (1).

System (1) is integrable on an open set  $\Omega$  of  $\mathbb{R}^2$  if there exists a non constant continuously differentiable function  $H : \Omega \rightarrow \mathbb{R}$ , called a first integral of the system on  $\Omega$ , which is constant on the trajectories of the system (1) contained in  $\Omega$ , i.e. if

$$\frac{dH(x, y)}{dt} = \frac{\partial H(x, y)}{\partial x} xF(x, y) + \frac{\partial H(x, y)}{\partial y} yG(x, y) \equiv 0 \quad \text{in the points of } \Omega.$$

Moreover,  $H = h$  is the general solution of this equation, where  $h$  is an arbitrary constant. It is well known that for differential systems defined on the plane  $\mathbb{R}^2$  the existence of a first integral determines their phase portrait [6].

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In this paper we are interested in studying the integrability and the periodic orbits of the 2-dimensional Kolmogorov systems of the form

$$\begin{cases} x' = x \left( P(x, y) + \sqrt{R(x, y)} \right), \\ y' = y \left( Q(x, y) + \sqrt{R(x, y)} \right), \end{cases} \quad (2)$$

where  $P(x, y)$ ,  $Q(x, y)$ ,  $R(x, y)$ , homogeneous polynomials of degree  $n$ ,  $n, m$  respectively.

We define the trigonometric functions  $f_1(\theta) = P(\cos \theta, \sin \theta) \cos^2 \theta + Q(\cos \theta, \sin \theta) \sin^2 \theta$ ,  $f_2(\theta) = \sqrt{R(\cos \theta, \sin \theta)}$  and  $f_3(\theta) = (Q(\cos \theta, \sin \theta) - P(\cos \theta, \sin \theta)) \cos \theta \sin \theta$ .

## 1. Main result

Our main result on the integrability and the periodic orbits of the Kolmogorov system (2) is the following.

**Theorem 1.** *For Kolmogorov system (2) the following statements hold.*

(a) *If  $f_3(\theta) \neq 0$ ,  $R(\cos \theta, \sin \theta) \geq 0$  and  $m \neq 2n$ , then system (2) has the first integral*

$$\begin{aligned} H(x, y) = & (x^2 + y^2)^{\frac{2n-m}{4}} \exp \left( \frac{m-2n}{2} \int_0^{\arctan \frac{y}{x}} A(\omega) d\omega \right) + \\ & + \left( \frac{m-2n}{2} \right) \int_0^{\arctan \frac{y}{x}} \exp \left( \frac{m-2n}{2} \int_0^w A(\omega) d\omega \right) B(w) dw, \end{aligned}$$

where  $A(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$ ,  $B(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$  and the curves which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as

$$\begin{aligned} x^2 + y^2 = & \left( h \exp \left( \frac{2n-m}{2} \int_0^{\arctan \frac{y}{x}} A(\omega) d\omega \right) + \right. \\ & + \frac{2n-m}{2} \exp \left( \frac{2n-m}{2} \int_0^{\arctan \frac{y}{x}} A(\omega) d\omega \right) \times \\ & \left. \times \int_0^{\arctan \frac{y}{x}} \exp \left( \frac{m-2n}{2} \int_0^w A(\omega) d\omega \right) B(w) dw \right)^{\frac{2}{2n-m}}, \end{aligned}$$

where  $h \in \mathbb{R}$ . Moreover, the system (2) has no a limit cycle.

(b) *If  $f_3(\theta) \neq 0$ ,  $R(\cos \theta, \sin \theta) \geq 0$  and  $m = 2n$ , then system (2) has the first integral*

$$H(x, y) = (x^2 + y^2)^{\frac{1}{2}} \exp \left( - \int_0^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right),$$

and the curves which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as

$$(x^2 + y^2)^{\frac{1}{2}} - h \exp \left( \int_0^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right) = 0,$$

where  $h \in \mathbb{R}$ . Moreover, the system (2) has no a limit cycle.

(c) If  $f_3(\theta) = 0$  for all  $\theta \in \mathbb{R}$  and  $R(\cos \theta, \sin \theta) \geq 0$ , then system (2) has the first integral  $H = \frac{y}{x}$  and the curves which are formed by the trajectories of the differential system (2), in Cartesian coordinates are written as  $y - hx = 0$ , where  $h \in \mathbb{R}$ . Moreover, the system (2) has no a limit cycle.

*Proof.* In order to prove our results we write the polynomial differential system (2) in polar coordinates  $(r, \theta)$ , defined by  $x = r \cos \theta$ , and  $y = r \sin \theta$ , then system (2) becomes

$$\begin{cases} r' = f_1(\theta) r^{n+1} + f_2(\theta) r^{\frac{1}{2}m+1}, \\ \theta' = f_3(\theta) r^n, \end{cases} \quad (3)$$

where the trigonometric functions  $f_1(\theta)$ ,  $f_2(\theta)$ ,  $f_3(\theta)$  are given in introduction,  $r' = \frac{dr}{dt}$  and  $\theta' = \frac{d\theta}{dt}$ .

If  $f_3(\theta) \neq 0$ ,  $R(\cos \theta, \sin \theta) \geq 0$  and  $n \neq 2m$ .

Taking as new independent variable the coordinate  $\theta$ , this differential system (3) writes

$$\frac{dr}{d\theta} = A(\theta) r + B(\theta) r^{\frac{m-2n+2}{2}}, \quad (4)$$

which is a Bernoulli equation. By introducing the standard change of variables  $\rho = r^{\frac{2n-m}{2}}$  we obtain the linear equation

$$\frac{d\rho}{d\theta} = \left( \frac{2n-m}{2} \right) (A(\theta) \rho + B(\theta)). \quad (5)$$

The general solution of linear equation (5) is

$$\begin{aligned} \rho(\theta) &= \exp\left(\frac{2n-m}{2} \int_0^\theta A(\omega) d\omega\right) \times \\ &\times \left( \alpha + \frac{2n-m}{2} \int_0^\theta \exp\left(\frac{m-2n}{2} \int_0^w A(\omega) d\omega\right) B(w) dw \right), \end{aligned}$$

where  $\alpha \in \mathbb{R}$ , then system (2) has the first integral

$$\begin{aligned} H(x, y) &= (x^2 + y^2)^{\frac{2n-m}{4}} \exp\left(\frac{m-2n}{2} \int_0^{\arctan \frac{y}{x}} A(\omega) d\omega\right) + \\ &+ \left(\frac{m-2n}{2}\right) \int_0^{\arctan \frac{y}{x}} \exp\left(\frac{m-2n}{2} \int_0^w A(\omega) d\omega\right) B(w) dw. \end{aligned}$$

Let  $\gamma$  be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let  $h_\gamma = H(\gamma)$ .

The curves  $H = h$  with  $h \in \mathbb{R}$ , which are formed by trajectories of the differential system (2), in Cartesian coordinates written as

$$\begin{aligned} x^2 + y^2 &= \left( h \exp\left(\frac{2n-m}{2} \int_0^{\arctan \frac{y}{x}} A(\omega) d\omega\right) + \right. \\ &+ \frac{2n-m}{2} \exp\left(\frac{2n-m}{2} \int_0^{\arctan \frac{y}{x}} A(\omega) d\omega\right) \times \\ &\times \left. \int_0^{\arctan \frac{y}{x}} \exp\left(\frac{m-2n}{2} \int_0^w A(\omega) d\omega\right) B(w) dw \right)^{\frac{2}{2n-m}}, \end{aligned}$$

where  $h \in \mathbb{R}$ .

Therefore the periodic orbit  $\gamma$  is contained in the curve

$$r(\theta) = \left( h_\gamma \exp \left( \frac{2n-m}{2} \int_0^\theta A(\omega) d\omega \right) + \frac{2n-m}{2} \exp \left( \frac{2n-m}{2} \int_0^\theta A(\omega) d\omega \right) \int_0^\theta \exp \left( \frac{m-2n}{2} \int_0^w A(\omega) d\omega \right) B(w) dw \right)^{\frac{2}{2n-m}}.$$

But this curve can not contain the periodic orbit  $\gamma$ , and consequently no limit cycle contained in one of the open quadrants because this curve has at most one point on every ray  $\theta = \theta^*$  for all  $\theta^* \in [0, 2\pi)$ . Hence statement (a) of Theorem 1 is proved.

Suppose now that  $f_3(\theta) \neq 0$ ,  $R(\cos \theta, \sin \theta) \geq 0$  and  $m = 2n$ .

Taking as new independent variable the coordinate  $\theta$ , this differential system (3) is written in the form

$$\frac{dr}{d\theta} = (A(\theta) + B(\theta)) r. \quad (6)$$

The general solution of equation (6) is

$$r(\theta) = \alpha \exp \left( \int_0^\theta (A(\omega) + B(\omega)) d\omega \right),$$

where  $\alpha \in \mathbb{R}$ , then system (2) has the first integral

$$H(x, y) = (x^2 + y^2)^{\frac{1}{2}} \exp \left( - \int_0^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right).$$

Let  $\gamma$  be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let  $h_\gamma = H(\gamma)$ .

The curves  $H = h$  with  $h \in \mathbb{R}$ , which are formed by trajectories of the differential system (2), in Cartesian coordinates are written as

$$(x^2 + y^2)^{\frac{1}{2}} - h \exp \left( \int_0^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right) = 0,$$

where  $h \in \mathbb{R}$ . Therefore the periodic orbit  $\gamma$  is contained in the curve

$$r(\theta) = h_\gamma \exp \left( \int_0^{\arctan \frac{y}{x}} (A(\omega) + B(\omega)) d\omega \right).$$

But this curve cannot contain the periodic orbit  $\gamma$ , and consequently no limit cycle contained in one of open quadrants because this curve at most have one point on every ray  $\theta = \theta^*$  for all  $\theta^* \in [0, 2\pi)$ . This completes the proof of statement (b) of Theorem 1.

Assume now that  $f_3(\theta) = 0$  for all  $\theta \in \mathbb{R}$  and  $R(\cos \theta, \sin \theta) \geq 0$ , then from system (3) it follows that  $\theta' = 0$ . So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence,  $\frac{y}{x}$  is a first integral of the system (2), then curves which are formed by the trajectories of the differential system (2), in Cartesian coordinates written as  $y - hx = 0$ , where  $h \in \mathbb{R}$ , since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, and consequently no limit cycle.

Hence statement (c) of Theorem 1 is proved.  $\square$

## Conclusion

The elementary method used in this paper seems to be fruitful to investigate more general planar differential systems of ODEs in order to obtain explicit expression for a first integral which characterizes its trajectories. This is one of the classical tools in the classification of all trajectories of dynamical systems.

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## О динамике одного класса систем Колмогорова

Рашид Букуша

*В этой статье мы характеризуем интегрируемость и отсутствие предельных циклов систем Колмогорова вида*

$$\begin{cases} x' = x \left( P(x, y) + \sqrt{R(x, y)} \right), \\ y' = y \left( Q(x, y) + \sqrt{R(x, y)} \right), \end{cases}$$

где  $P(x, y)$ ,  $Q(x, y)$ ,  $R(x, y)$  — однородные многочлены степени  $n$ ,  $n$ ,  $m$  соответственно.

*Ключевые слова:* система Колмогорова, первый интеграл, периодические орбиты, предельный цикл.